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2010 J. Phys. A: Math. Theor. 43 145202

(http://iopscience.iop.org/1751-8121/43/14/145202)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **43** (2010) 145202 (18pp) doi:10.1088/1751-8113/43/14/145202

Solvable Lie algebras with an $\mathbb{N}$-graded nilradical of maximal nilpotency degree and their invariants

R Campoamor-Stursberg

I.M.I. and Dpto. Geometría y Topología, Universidad Complutense de Madrid, Plaza de Ciencias, 3 E-28040 Madrid, Spain

E-mail: rutwig@pdi.ucm.es

Received 24 November 2009, in final form 9 February 2010
Published 16 March 2010
Online at stacks.iop.org/JPhysA/43/145202

Abstract
The class of solvable Lie algebras with an $\mathbb{N}$-graded nilradical of maximal nilpotency index is classified. It is shown that such solvable extensions are unique up to isomorphism. The generalized Casimir invariants for the $\mathbb{N}$-graded nilradicals and their associated solvable extensions are computed by the method of moving frames.

PACS numbers: 02.20.Qs, 02.40.Sv, 03.65.Fd

1. Introduction

While semisimple Lie algebras have played a major role in applications to atomic, nuclear and elementary particle physics, and constitute nowadays a standard tool in these disciplines [1], solvable Lie algebras were recognized much later to be of interest in physics. Although used in the classification of gravitational fields [2], their common use has only started in recent decades, where they have been applied systematically to (completely) integrable systems, in the formulation of non-Abelian gauge theories or their applications in quantum gravity and string theories in the low energy limit [3–5]. In contrast to the semisimple case, solvable Lie algebras over the real field $\mathbb{R}$ have been classified only in low dimensions, due to the absence of global structural properties, as well as the existence of parameterized families [6, 7]. An important effort in this direction was made by the Kazan school, mainly by Morozov [8] and his collaborators, who classified real solvable Lie algebras up to dimension 6 [9, 10] (with one case left that was later solved in [11]) and also analyzed the seven-dimensional case under certain assumptions [12]. It is worthy to mention that these lists were used in [2], as well as in other applications to cosmology [13]. In any case, a global classification beyond dimension 6 seems not possible, and the analysis is focused on certain specific classes of algebras of potential value in applications (see e.g. [14] and references therein). Recently, the classification of large classes of solvable Lie algebras has recovered interest in the context of integrability of Hamiltonian systems, by means of the so-called coalgebra symmetry formalism [15, 16].

For the latter, the exact knowledge of the generalized Casimir invariants is of fundamental importance. The structure of these invariants in solvable algebras differs radically from the semisimple case, where they are well known [1]. In particular, for solvable algebras there is no intrinsic quantity that determines the number of invariants, and it is not unusual to find non-rational invariants [6]. Invariants of solvable Lie algebras have been obtained in low dimensions [6, 7, 17–20], as well as for certain families in arbitrary dimension [14, 21–28]. All these results provide large classes that allow the construction of new Hamiltonian systems with N-degrees of freedom and nonlinear interactions [16].

In this work, we consider the class of nilpotent Lie algebras with maximal nilpotency index possessing an $\mathbb{N}$-grading. This condition is a relaxation of a natural grading and provides a larger number of isomorphism classes [29]. It turns out that this class contains the nilpotent algebras considered in [25] and [14], and also provides parameterized families in dimension $n \leqslant 11$. Together with [25, 26] and [14], this work finishes the analysis of the most relevant classes of solvable Lie algebras having nilradicals of maximal degree of nilpotency. It is shown that the solvable extensions of these $\mathbb{N}$-graded nilradicals are generically characterized by the derivation induced by the grading. This in particular implies that there is only one class of solvable Lie algebras extending these nilradicals, as happened with the algebras studied in [14]. Moreover, the generalized Casimir invariants of these algebras and their solvable extensions are computed by means of the method of moving frames.

This paper is structured as follows. Sections 2 and 3 present the main facts concerning the computation of invariants by means of the moving frames method, as well as the classification of $\mathbb{N}$-graded Lie algebras of maximal degree of nilpotency. In section 4, their Casimir operators are explicitly obtained. Sections 5 and 6 are devoted to the classification of solvable extensions and the invariants of these. Finally, the conclusions present some potential applications of the results obtained and a short overview of the classes of solvable algebras analyzed in the literature.

We apply the Einstein convention and usual notations for tensor algebra. By indecomposable Lie algebras we mean algebras that do not split into a direct sum of ideals. Unless otherwise stated, any Lie algebra considered in this work is defined over the field $\mathbb{R}$.

2. Invariants of Lie algebras and moving frames

The invariant operators of the coadjoint representation of Lie algebras provide important information on a physical system, like quantum numbers, energy spectra or the existence of invariant forms. Polynomial invariants are traditionally called Casimir invariants and occur for semisimple and nilpotent Lie algebras. More generally, algebraic Lie algebras always admit invariants that are rational. For non-algebraic Lie algebras, specially for those that are solvable, we find rational or even transcendental invariants. These find wide applications in (classical) integrable systems [3]. In fact, the algorithm usually applied to compute these invariants [30, 31], based on a system of linear first-order partial differential equations, does not exclude the existence of irrational invariants, nor is there any physical reason for the invariants to be polynomials. In analogy with the classical Casimir operators, nonpolynomial invariants are called generalized Casimir invariants.

Let $\{X_1, \ldots, X_n\}$ be a basis of $\mathfrak{g}$ and $\{C_{ij}^k\}$ be the structure constants over this basis. We consider the representation of $\mathfrak{g}$ in the space $C^\infty(\mathfrak{g}^*)$ given by

$$\widehat{X}_i = -C_{ij}^k x_k \partial_{x_j}, \tag{1}$$

where $[X_i, X_j] = C_{ij}^k X_k$ $(1 \leqslant i < j \leqslant n,\ 1 \leqslant k \leqslant n)$. This representation is easily seen to satisfy the brackets $[\widehat{X}_i, \widehat{X}_j] = C_{ij}^k \widehat{X}_k$. The invariants are functions on the generators $F(X_1, \ldots, X_n)$ of $\mathfrak{g}$ such that

$$[X_i, F(X_1, \ldots, X_n)] = 0 \tag{2}$$

and are found by solving the system of linear first-order partial differential equations:

$$\widehat{X}_i F(x_1, \ldots, x_n) = -C_{ij}^k x_k \partial_{x_j} F(x_1, \ldots, x_n) = 0, \qquad 1 \leqslant i \leqslant n, \tag{3}$$

and then replacing the variables x_i by the corresponding generator X_i (possibly after symmetrizing). A maximal set of functionally independent solutions of (3) will be called a fundamental set of invariants. The cardinal $\mathcal{N}(\mathfrak{g})$ of such a set can be described in terms of the dimension and a certain matrix associated with the commutator table. More specifically, denote by $A(\mathfrak{g})$ the matrix representing the commutator table of $\mathfrak{g}$ over a given basis, i.e.

$$A(\mathfrak{g}) = \left(C_{ij}^k x_k\right). \tag{4}$$

Such a matrix has necessarily even rank by antisymmetry. Then $\mathcal{N}(\mathfrak{g})$ is given by

$$\mathcal{N}(\mathfrak{g}) = \dim\ \mathfrak{g} - \operatorname{rank}\left(C_{ij}^k x_k\right). \tag{5}$$

This formula was first given in [31]. With respect to the number of independent Casimir operators of $\mathfrak{g}$, formula (5) is merely an upper bound. For high-dimensional Lie algebras, it is sometimes convenient to work with the analogue of formula (5) in terms of differential forms. Let $\mathcal{L}(\mathfrak{g}) = \mathbb{R}\{d\omega_i\}_{1 \leqslant i \leqslant \dim \mathfrak{g}}$ be the linear subspace of $\bigwedge^2 \mathfrak{g}^*$ generated by the Maurer–Cartan forms $d\omega_i$ of $\mathfrak{g}$. If $\omega = a^i d\omega_i$ $(a^i \in \mathbb{R})$ is a generic element of $\mathcal{L}(\mathfrak{g})$, there always exists an integer $j_0(\omega) \in \mathbb{N}$ such that

$$\bigwedge^{j_0(\omega)} \omega \neq 0, \qquad \bigwedge^{j_0(\omega)+1} \omega \equiv 0. \tag{6}$$

This equation shows that $r(\omega) = 2 j_0(\omega)$ is the rank of the 2-form ω. We now define

$$j_0(\mathfrak{g}) = \max\{j_0(\omega) \mid \omega \in \mathcal{L}(\mathfrak{g})\}. \tag{7}$$

The quantity $j_0(\mathfrak{g})$, which depends only on the structure of $\mathfrak{g}$, constitutes a numerical invariant of $\mathfrak{g}$ [32]. The number of invariants follows from the expression

$$\mathcal{N}(\mathfrak{g}) = \dim \mathfrak{g} - 2 j_0(\mathfrak{g}). \tag{8}$$

A second method of calculating invariants of group actions that has proven to be of great interest is based on the Cartan theory of moving frames [33]. A recent reformulation of this procedure [34] provides an algebraic algorithm for computing the generalized Casimir operators. We briefly recall here the main features of the moving frame method, according to the recent formulation of [34]. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and let $\mathfrak{g}^*$ denote its dual. The coadjoint representation of G is given by the map $\mathrm{Ad}^* : G \to GL(\mathfrak{g}^*)$ defined by

$$\langle \mathrm{Ad}_g^* x, u\rangle = \langle x, \mathrm{Ad}_{g^{-1}} u\rangle \qquad x \in \mathfrak{g}^*, u \in \mathfrak{g},$$

where Ad denotes the usual adjoint representation of G. A function $F \in C^\infty(\mathfrak{g}^*)$ is an invariant of Ad_G^* if $F(\mathrm{Ad}_g^* x) = F(x)$ for all $g \in G$ and $x \in \mathfrak{g}^*$. The set of invariants of Ad_G^* is denoted by $\mathrm{Inv}(\mathrm{Ad}_G^*)$. If $\mathcal{G} = \mathrm{Ad}_G^* \times \mathfrak{g}^*$ denotes the trivial left principal Ad_G^*-bundle over $\mathfrak{g}^*$, then the right regularization $\widehat{R}$ of the coadjoint action of G on $\mathfrak{g}^*$ is the diagonal action of Ad_G^* on $\mathcal{G} = \mathrm{Ad}_G^* \times \mathfrak{g}^*$, explicitly given by the map $\widehat{R}_g(\mathrm{Ad}_h^*, x) = (\mathrm{Ad}_h^* \cdot \mathrm{Ad}_{g^{-1}}^*, \mathrm{Ad}_g^* x)$, $g, h \in G$, $x \in \mathfrak{g}^*$ [33]. We call $\widehat{R}_g$ the lifted coadjoint action of G. It pulls back to the coadjoint action on $\mathfrak{g}^*$ via the Ad_G^*-equivariant projection $\pi_{\mathfrak{g}^*} : \mathcal{G} \to \mathfrak{g}^*$. A lifted invariant of Ad^*_G is therefore a local function from $\mathcal{G}$ to a manifold invariant with respect to the lifted coadjoint action. The function $\mathcal{I} : \mathcal{G} \to \mathfrak{g}^*$ given by $\mathcal{I} = \mathcal{I}(\mathrm{Ad}^*_g, x) = \mathrm{Ad}^*_g x$ is called a fundamental lifted invariant of Ad^*_G.

Based on this method, a simplified algebraic algorithm to compute invariants was proposed in [34] and used successfully in [27, 28, 35]. The four steps of this reformulation are as follows.

(i) Determination of the generic matrix $B(\theta)$ of an inner automorphism of $\mathfrak{g}$ on a given basis, where $\theta = (\theta_1, \ldots, \theta_r)$ denotes coordinates of Ad_G, and r is the codimension of the center $Z(\mathfrak{g})$ in $\mathfrak{g}$.

(ii) Representation of the fundamental lifted invariants: the explicit form of these fundamental lifted invariants $\mathcal{I} = (\mathcal{I}_1, \ldots, \mathcal{I}_n)$ of Ad^*_G in the coordinates $(\theta, \check{x})$ of $\mathrm{Ad}^*_G \times \mathfrak{g}^*$ is $\mathcal{I} = \check{x} \cdot B(\theta)$, i.e. $(\mathcal{I}_1, \ldots, \mathcal{I}_n) = (x_1, \ldots, x_n) \cdot B(\theta_1, \ldots, \theta_r)$.

(iii) Elimination of parameters: a maximal number $2j_0(\mathfrak{g})$ of lifted invariants $\mathcal{I}_{j_1}, \ldots, \mathcal{I}_{j_{2j_0(\mathfrak{g})}}$, constants $c_1, \ldots, c_{2j_0(\mathfrak{g})}$ and group parameters $\theta_{k_1}, \ldots, \theta_{k_{2j_0(\mathfrak{g})}}$ such that the equations $\mathcal{I}_{j_1} = c_1, \ldots, \mathcal{I}_{j_{2j_0(\mathfrak{g})}} = c_{2j_0(\mathfrak{g})}$ are solvable with respect to $\theta_{k_1}, \ldots, \theta_{k_{2j_0(\mathfrak{g})}}$. Substitution of the found values of $\theta_{k_1}, \ldots, \theta_{k_{2j_0}(\mathfrak{g})}$ into the lifted invariants provides $\mathcal{N}(\mathfrak{g})$ functions $F^l(x_1, \ldots, x_n)$ that are independent on the θ's.

(iv) Symmetrization: the latter functions $F^l(x_1, \ldots, x_n)$ are symmetrized to $\mathrm{Sym}(F^l(X_1, \ldots, X_n))$ by means of the symmetrization operator $\mathrm{Sym}(x_1 \ldots x_p) = \frac{1}{p!} \sum_{\sigma \in S_p} X_{\sigma(1)} \ldots X_{\sigma(p)}$.

3. $\mathbb{N}$-graded Lie algebras with maximal degree of nilpotency

To any Lie algebra $\mathfrak{g}$ we can naturally associate various recursive series of ideals:

$$D^0\mathfrak{g} = \mathfrak{g} \supset D^1\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \supset \cdots \supset D^k\mathfrak{g} = [D^{k-1}\mathfrak{g}, D^{k-1}\mathfrak{g}] \supset \cdots \tag{9}$$

$$C^0\mathfrak{g} = \mathfrak{g} \supset C^1\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \supset \cdots \supset C^k\mathfrak{g} = [\mathfrak{g}, C^{k-1}\mathfrak{g}] \supset \cdots \tag{10}$$

called respectively the derived and central descending sequence. Solvability is given whenever the derived series is finite, i.e. if there exists a k such that $D^k\mathfrak{g} = 0$, and nilpotency whenever the central descending sequence is finite, i.e. if $C^k\mathfrak{g} = 0$ for some k. The dimensions of the subalgebras in both series provide numerical invariants of the Lie algebra. We use the notation DS and CDS for the dimension sequences of the descending and central descending sequences, respectively.

Starting from the central descending sequence, we can associate a graded Lie algebra $\mathfrak{gr}(\mathfrak{g})$ with $\mathfrak{g}$, which is defined by $\mathfrak{gr}(\mathfrak{g}) := \sum_{i \geqslant 0} \mathfrak{g}_{i+1}$, where

$$\mathfrak{g}_{i+1} := \frac{C^i\mathfrak{g}}{C^{i+1}\mathfrak{g}}, \qquad i \geqslant 0. \tag{11}$$

It satisfies the condition

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}, \qquad 1 \leqslant i, j. \tag{12}$$

A Lie algebra is said naturally graded if $\mathfrak{g}$ and $\mathfrak{gr}(\mathfrak{g})$ are isomorphic Lie algebras.

It is not difficult to see that the graded algebra associated with any Lie algebra $\mathfrak{g}$ always arises as a contraction of the latter. In this sense, (naturally) graded algebras constitute a certain basic structure that allows the analysis of non-graded structures by means of deformation theory [37].

For naturally graded nilpotent Lie algebras of maximal nilindex, all solvable extensions and their invariants have been classified and studied in [25] and [26]. Since the natural grading is an excessively restrictive condition for nilpotent algebras with maximal nilindex (see [26] and references therein), it makes sense to relax it, still requiring some underlying graded structure. One such possibility is the $\mathbb{N}$-grading, which is deeply related to the structure of some external derivations. For the case of degree of nilpotency $n-1$, it provides various interesting families of algebras in arbitrary dimension, as well as some parameterized algebras in low dimensions. Recall that an n-dimensional nilpotent Lie algebra $\mathfrak{n} = \bigoplus_{i\geqslant 1} \mathfrak{n}_i$ is called $\mathbb{N}$-graded if $[\mathfrak{n}_i, \mathfrak{n}_j] = \mathfrak{n}_{i+j}$ for $1 \leqslant i, j$. If we require in addition that the algebra is of maximal degree of nilpotency $n-1$, then $\dim \mathfrak{n}_1 = 2$, $\dim \mathfrak{n}_i = 1$ for $i = 2, \ldots, n$ and $[\mathfrak{n}_1, \mathfrak{n}_i] = \mathfrak{n}_{i+1}$ for $i \geqslant 2$. Algebras of this type have appeared in some contexts, such as the classification of vector fields on the line and the analysis of rigid algebras [38, 39]. The requirement on the nilpotency index implies further that the central series is given by $\mathrm{CDS} = [n, n-2, n-3, \ldots, 2, 1, 0]$

Theorem 1. *Any $\mathbb{N}$-graded nilpotent Lie algebra $\mathfrak{n}$ of maximal degree of nilpotency is isomorphic to one of the following algebras:*

(i) $\mathfrak{n}_{n,1}$ $(n \geqslant 3)$, $\quad DS = [n, n-2, 0]$:
$$[X_1, X_j] = X_{j+1}, \quad 2 \leqslant j \leqslant n-1.$$
(ii) $\mathfrak{n}_{n,2}$ $(n \geqslant 5)$, $\quad DS = [n, n-2, 0]$:
$$[X_1, X_j] = X_{j+1}, \quad 2 \leqslant j \leqslant n-1,$$
$$[X_2, X_j] = X_{j+2}, \quad 3 \leqslant j \leqslant n-2.$$
(iii) $\mathfrak{n}_{n,3}$ $(n \geqslant 12)$, $\quad DS = [n, n-2, n+2-2^3, n+2-2^4, \ldots, n+2-2^{j+1}, 0]$, *$j$ being the largest integer such that $n+2-2^{j+1} > 0$* :
$$[X_i, X_j] = (j-i)X_{i+j}, \quad 1 \leqslant i < j \leqslant n-1.$$
(iv) $\mathfrak{n}_{n,4}$ $(n = 2m+1 \geqslant 7)$, $\quad DS = [n, n-2, 1, 0]$:
$$[X_1, X_j] = X_{j+1}, \quad 2 \leqslant j \leqslant n-1,$$
$$[X_i, X_{n-i}] = (-1)^i X_n, \quad 2 \leqslant i \leqslant m.$$
(v) $\mathfrak{n}_{n,5}$ $(n = 2m \geqslant 8)$, $\quad DS = [n, n-2, 2, 0]$:
$$[X_1, X_j] = X_{j+1}, \quad 2 \leqslant j \leqslant n-1,$$
$$[X_i, X_{n-i-1}] = (-1)^{i+1} X_{n-1}, \quad 2 \leqslant i \leqslant m-1,$$
$$[X_i, X_{n-i}] = (-1)^{i+1}(m-i) X_n, \quad 2 \leqslant i \leqslant m-1.$$
(vi) $\mathfrak{n}_{n,6}$ $(n = 2m+3 \geqslant 9)$, $\quad DS = [n, n-2, 3, 0]$:
$$[X_1, X_j] = X_{j+1}, \quad 2 \leqslant j \leqslant n-1,$$
$$[X_i, X_{n-i-2}] = (-1)^i X_{n-2}, \quad 2 \leqslant i \leqslant m,$$
$$[X_i, X_{n-i-1}] = (-1)^i (m-i) X_{n-1}, \quad 2 \leqslant i \leqslant m,$$
$$[X_i, X_{n-i}] = (-1)^{i+1} \tfrac{(i-2)}{2}(2m-i-1) X_n, \quad 3 \leqslant i \leqslant m+1.$$
(vii) $\mathfrak{n}_{7,\alpha}$, $\quad DS = [7, 5, 1, 0]$:
$$[X_1, X_j] = X_{j+1}\ (2 \leqslant j \leqslant 6), \quad [X_2, X_3] = (2+\alpha)X_5,$$
$$[X_2, X_4] = (2+\alpha)X_6, \quad [X_2, X_5] = (1+\alpha)X_7, \quad [X_3, X_4] = X_7.$$
(viii) $\mathfrak{n}_{8,\alpha}$, $\quad DS = [8, 6, 2, 0]$:
$$[X_1, X_j] = X_{j+1}\ (2 \leqslant j \leqslant 7), \quad [X_2, X_3] = (2+\alpha)X_5,$$
$$[X_2, X_4] = (2+\alpha)X_6, \quad [X_2, X_5] = (1+\alpha)X_7, \quad [X_2, X_6] = \alpha X_8,$$
$$[X_3, X_4] = X_7, \quad [X_3, X_5] = X_8.$$
(ix) $\mathfrak{n}_{9,\alpha}$ $\left(\alpha \neq -\tfrac{5}{2}\right)$, $\quad DS = [9, 7, 3, 0]$
$$[X_1, X_j] = X_{j+1}\ (2 \leqslant j \leqslant 8), \quad [X_2, X_3] = (2+\alpha)X_5, \quad [X_2, X_4] = (2+\alpha)X_6,$$
$$[X_2, X_5] = (1+\alpha)X_7, \quad [X_2, X_6] = \alpha X_8, \quad [X_2, X_7] = \tfrac{2\alpha^2+3\alpha-2}{2\alpha+5} X_9,$$
$$[X_3, X_4] = X_7, \quad [X_3, X_5] = X_8, \quad [X_3, X_6] = \tfrac{2\alpha-1}{2\alpha+5} X_9, \quad [X_4, X_5] = \tfrac{3}{2\alpha+5} X_9.$$
(x) $\mathfrak{n}_{10,\alpha}$ $\left(\alpha \neq -\tfrac{5}{2}\right)$, $\quad DS = [10, 8, 4, 0]$:

$$[X_1, X_j] = X_{j+1}\ (2 \leqslant j \leqslant 9), \quad [X_2, X_3] = (2+\alpha)X_5, \quad [X_2, X_4] = (2+\alpha)X_6,$$
$$[X_2, X_5] = (1+\alpha)X_7, \quad [X_2, X_6] = \alpha X_8, \quad [X_2, X_7] = \tfrac{2\alpha^2+3\alpha-2}{2\alpha+5}X_9,$$
$$[X_2, X_8] = \tfrac{2\alpha^2+\alpha+1}{2\alpha+5}X_{10}, \quad [X_3, X_4] = X_7, \quad [X_3, X_5] = X_8,$$
$$[X_3, X_6] = \tfrac{2\alpha-1}{2\alpha+5}X_9, \quad [X_3, X_7] = \tfrac{2\alpha-1}{2\alpha+5}X_{10}, \quad [X_4, X_5] = \tfrac{3}{2\alpha+5}X_9,$$
$$[X_4, X_6] = \tfrac{3}{2\alpha+5}X_{10},$$

(xi) $\mathfrak{n}_{11,\alpha}$ $\left(\alpha \neq -3, -\frac{5}{2}, -1\right)$, $\quad DS = [11, 9, 5, 0]$:

$$[X_1, X_j] = X_{j+1}\ (2 \leqslant j \leqslant 10), \quad [X_2, X_3] = (2+\alpha)X_5,$$
$$[X_2, X_4] = (2+\alpha)X_6, \quad [X_2, X_5] = (1+\alpha)X_7, \quad [X_2, X_6] = \alpha X_8,$$
$$[X_2, X_7] = \tfrac{2\alpha^2+3\alpha-2}{2\alpha+5}X_9, \quad [X_2, X_8] = \tfrac{2\alpha^2+\alpha+1}{2\alpha+5}X_{10}, \quad [X_2, X_9] = \tfrac{2\alpha^3+2\alpha^2+3}{2(\alpha^2+4\alpha+3)}X_{11},$$
$$[X_3, X_4] = X_7, \quad [X_3, X_5] = X_8, \quad [X_3, X_6] = \tfrac{2\alpha-1}{2\alpha+5}X_9,$$
$$[X_3, X_7] = \tfrac{2\alpha-1}{2\alpha+5}X_{10}, \quad [X_3, X_8] = \tfrac{4\alpha^3+8\alpha^2-8\alpha-21}{2(\alpha^2+4\alpha+3)(2\alpha+5)}X_{11}, \quad [X_4, X_5] = \tfrac{3}{2\alpha+5}X_9,$$
$$[X_4, X_6] = \tfrac{3}{2\alpha+5}X_{10}, \quad [X_4, X_7] = \tfrac{3(2\alpha^2+4\alpha+5)}{2(\alpha^2+4\alpha+3)(2\alpha+5)}X_{11},$$
$$[X_5, X_6] = \tfrac{3(4\alpha+1)}{2(\alpha^2+4\alpha+3)(2\alpha+5)}X_{11}.$$

A proof of this result can be found in [29]. It actually enlarges previous analysis of this class of algebras made in different contexts. We observe that $\mathfrak{n}_{n,1}$ is actually naturally graded and coincides with the nilradical studied in [25]. The Lie algebra $\mathfrak{n}_{n,2}$ was the subject of [14] and has also played an important role in the analysis of stable Lie algebras with nontrivial cohomology [38]. The algebra $\mathfrak{n}_{n,3}$ corresponds to the Lie algebra of polynomial vector fields on the line [39] and is easily seen to be related with the finite-dimensional quotients of the Virasoro algebra [40]. The algebras $\mathfrak{n}_{n,4}$, $\mathfrak{n}_{n,5}$ and $\mathfrak{n}_{n,6}$ were already studied in [41] in the context of stability theory, while the explicit classification of the parameterized families was first considered in [29].

4. Generalized Casimir invariants of $\mathbb{N}$-graded nilradicals

In this section we determine the Casimir invariants of the $\mathbb{N}$-graded Lie algebras of proposition 1, with the exception of the two first algebras $\mathfrak{n}_{n,1}$ and $\mathfrak{n}_{n,2}$, which have been exhaustively analyzed in [25] and [14], respectively. It is convenient to separate the algebras in arbitrary dimension from the parameterized families in low dimension, as these present a certain number of particular cases that must be studied separately.

Lemma 1. *Let* $\mathfrak{n}$ *be isomorphic to* $\mathfrak{n}_{n,k}$ *for* $k = 3, 4, 5, 6$. *Then the following identities hold:*

(i) $\mathcal{N}(\mathfrak{n}_{n,3}) = \begin{cases} 1 & \text{for any odd } n \geqslant 13 \\ 2 & \text{for any even } n \geqslant 12 \end{cases}$,
(ii) $\mathcal{N}(\mathfrak{n}_{n,4}) = 1$ *for any* $n = 2m+1 \geqslant 7$,
(iii) $\mathcal{N}(\mathfrak{n}_{n,5}) = 2$ *for any* $n = 2m \geqslant 8$,
(iv) $\mathcal{N}(\mathfrak{n}_{n,6}) = 3$ *for any* $n = 2m+1 \geqslant 9$.

The proof of these statements follows at once considering the Maurer–Cartan equations for each algebra. For example, for $\mathfrak{n}_{n,3}$ we obtain the equations

$$\mathrm{d}\omega_1 = \mathrm{d}\omega_2 = 0,$$
$$\mathrm{d}\omega_k = \textstyle\sum_{i+j=k}(j-i)\omega_i \wedge \omega_j, \qquad k \geqslant 3.$$

In particular, the 2-form corresponding to the center generator X_n is given by

$$\mathrm{d}\omega_n = \sum_{j=1}^{[\frac{n-1}{2}]}(n-2j)\omega_j \wedge \omega_{n-j}.$$

Considering the wedge products it is straightforward to verify that $\bigwedge^{[\frac{n-1}{2}]} \mathrm{d}\omega_n \neq 0$, thus $j_0(\mathfrak{n}_{n,3}) = [\frac{n-1}{2}]$. Using formula (5) the number of invariants is given by

$$\mathcal{N}(\mathfrak{n}_{n,3}) = \begin{cases} 2m - 2\left[m - \frac{1}{2}\right] = 2, & n = 2m \\ 2m + 1 - 2m = 1, & n = 2m + 1 \end{cases}.$$

Since these algebras are nilpotent, one of the Casimir operators is always given by the center generator X_n. We thus only need to compute the non-central invariants. As is not unusual for nilpotent algebras, in spite of the apparent simplicity of their brackets, the non-central invariants are quite complicated polynomials.

Proposition 1. *Let* $\mathfrak{n}$ *be isomorphic to* $\mathfrak{n}_{n,k}$ *for* $k = 3, 5, 6$.

(i) The non-central Casimir of $\mathfrak{n}_{n,3}$ *(*$n = 2m \geqslant 12$*) is given by*

$$\begin{aligned} C_{2m,m} &= x_m x_{2m}^{m-1} + \sum_{l=2}^{m-3} \sum_{j=0}^{l} \sum_{p=0}^{j} \frac{\Gamma(l+\frac{1}{2})(-1)^l}{(l-j)!(1+j-p)!p!\sqrt{\pi}} x_{2m-1}^{l-j} x_{m+a}^{1+j-p} x_{m+b}^{p} x_{2m}^{m-l-1} \\ &\quad - \frac{1}{2} \sum_{j=1}^{[\frac{m}{2}]} \left(x_{m+j} x_{2m-j} + \frac{1+(-1)^m}{4} x_{\frac{3m}{2}}^2 \right) x_{2m}^{m-2} - \frac{(-1)^{m-2}}{\sqrt{\pi}} \frac{\Gamma(m-\frac{1}{2})}{\Gamma(m+1)} x_{2m-1}^{m} \\ &\quad + \frac{(-1)^{m-2}}{\sqrt{\pi}} \left(\frac{\Gamma(m-\frac{3}{2})}{\Gamma(m-1)} x_{2m} x_{2m-1}^{m-2} x_{2m-2} \right), \end{aligned} \tag{13}$$

where the indices a, b *are obtained from the constraint*

$$(1 + j - p)(m + a) + (m + b)p = (2j + 1)m + (l - j). \tag{14}$$

(ii) The non-central Casimir of $\mathfrak{n}_{n,5}$ *(*$n = 2m \geqslant 8$*) is given by*

$$C_{2m,m} = x_{2m-1}^{m} + \sum_{j=1}^{m-1} (-1)^j \frac{m(m-2)!}{(m+1-j)!} x_{2m-1-j} x_{2m-1}^{m-j-1} x_{2m}^{j}. \tag{15}$$

(iii) The non-central Casimir invariants of $\mathfrak{n}_{n,6}$ *(*$n = 2m + 3 \geqslant 9$*) are given by* $C_{2m+3,2} = 2x_{2m+1}x_{2m+3} - x_{2m+2}^2$ *and*

$$C_{2m+3,2m+1} = \sum_{j=-1}^{m-1} \kappa_{m,j} x_{2m+1}^{j+1} x_{2m+2}^{2m-1-2j} x_{2m+3}^{j+1} + \sum_{l=1}^{m} \Phi_{m+1-l}, \tag{16}$$

where Φ_{m+1-l} *is defined by*

$$\kappa_{m,m-1} x_{2m+3}^{m+l} \sum_{j=l}^{m} \frac{(-1)^{j+l-1}(m-1)!}{(l-1)!(m-j)!} \prod_{k=2}^{l} \left(\frac{j(j-1) - (k-1)(k-2)}{2} \right) x_{2m+2-l-j} x_{2m+1}^{m-j} x_{2m+2}^{j-l}$$

and the coefficients $\kappa_{m,j}$ *are given by*

$$\kappa_{m,j} = \frac{(-1)^{m+2j+1} 2^{j+4} \Gamma(j + \frac{1}{2} - m)}{(j+1)!\Gamma(-m - \frac{1}{2})} \tag{17}$$

for any $j = -1, \ldots, m - 1$.

The proof follows by application of the method of moving frames in its algebraic reformulation of [34], combined with a recursion argument. We exemplify the procedure for the Lie algebras $\mathfrak{n}_{n,5}$, the remaining cases being solved using an analogous argument. The lifted invariants are obtained solving the system

$$[J_k] = [x_1, \ldots, x_{2m}] \prod_{i=1}^{2m} \exp\,(\mathrm{ad}X_i\theta_i) \tag{18}$$

for the variables θ_i, i.e. obtaining $\mathcal{N}(\mathfrak{n})$ functions that do not depend on the parameters θ_i.

Using the transitivity of matrix multiplication, we can rewrite the previous equation as

$$[J_k] = ([x_1, \ldots, x_{2m}] \exp\,(\mathrm{ad}X_1\theta_1)) \prod_{i\geqslant 2} \exp\,(\mathrm{ad}X_i\theta_i).$$

The vector $A = [x_1, \ldots, x_{2m}] \exp\,(\mathrm{ad}X_1\theta_1)$ is explicitly given by

$$\left[x_1, x_2 + \sum_{l=3}^{2m} \alpha_{l,2}x_l, \ldots, x_k + \sum_{l=k+1}^{2m} \alpha_{l,k}x_l, \ldots, x_{2m-1} + \alpha_{2m,2m-1}x_{2m}, x_{2m}\right],$$

where the coefficients $\alpha_{l,k}$ are determined by

$$\alpha_{j,k} = \begin{cases} 1, & j = k \\ \dfrac{1}{(j-k)!}\theta_1^{j-k}, & k \geqslant 2, \quad j \geqslant k+1\ . \\ 0, & \text{otherwise} \end{cases}$$

On the other hand, the columns Δ_k of the matrix $B = \prod_{i\geqslant 2} \exp\,(\mathrm{ad}X_i\theta_i)$ are given by

$$\Delta_1 = [1, 0, -\theta_2, \ldots, -\theta_{2m-3}, P_1(\theta), P_2(\theta)]^T,$$

$P_1(\theta)$ and $P_2(\theta)$ being the polynomials

$$P_1(\theta) = \sum_{k=2}^{m-2} \theta_k\theta_{2m-2-k} + \frac{(-1)^{m-1}}{2}\theta_{m+1}^2 - \theta_{2m-2},$$

$$P_2(\theta) = \sum_{k=2}^{m-2} (-1)^k(m-k)\theta_k\theta_{2m-1-k} - \theta_{2m-1},$$

$$\begin{aligned}
&\Delta_k = [0, \ldots, \delta_k^j, \ldots, (-1)^k\theta_{2m-1-k}, (-1)^k(m-k)\theta_{2m-k}]^T, \quad 2 \leqslant k \leqslant m-1,\\
&\Delta_m = [0, \ldots, \delta_m^j, \ldots, (-1)^m\theta_{m-1}, 0]^T,\\
&\Delta_{m+k} = [0, \ldots, \delta_{m+k}^j, \ldots, (-1)^{m+k}\theta_{m-1-k}, -(-1)^{m+k}k\theta_{m-k}]^T, \quad 1 \leqslant k \leqslant m-3,\\
&\Delta_{2m-2} = [0, \ldots, 0, \ldots, 0, 1, 0, -(m-2)\theta_2]^T,\\
&\Delta_{2m-1} = [0, \ldots, 0, \ldots, 1, 0]^T, \quad \Delta_{2m} = [0, \ldots, 0, \ldots, 0, 1]^T.
\end{aligned}$$

With these notations, the kth lifted invariant J_k is given by the product

$$J_k = A\Delta_k.$$

Developing this product for $k = 1, \ldots, n$ we get the system

$$J_1 = x_1 - \sum_{k=3}^{2m-2} \theta_{k-1}\left(x_k + \sum_{l=k+1}^{2m} \alpha_{l,k}x_l\right) + (x_{2m-1} + \alpha_{2m,2m-1}x_{2m})P_1(\theta) + P_2(\theta)x_{2m},$$

$$J_k = x_k + \sum_{l=k+1}^{2m} \alpha_{l,k}x_l + (-1)^k(x_{2m-1} + \alpha_{2m,2m-1}x_{2m})\theta_{2m-1-k} + (-1)^k(m-k)\theta_{2m-k}x_{2m}$$

Table 1. $\mathcal{N}(\mathfrak{n}_{j,\alpha})$ for parameterized families having more than one invariant.

	$\mathfrak{n}_{7,-1}$	$\mathfrak{n}_{8,\alpha}$	$\mathfrak{n}_{9,\alpha}$	$\mathfrak{n}_{10,\alpha}$	$\mathfrak{n}_{10,\alpha}$	$\mathfrak{n}_{11,\alpha}$[a]
α			$-2,-1,\frac{1}{2}$	$\neq -1,\frac{1}{2}$	$-1,\frac{1}{2}$	$-\frac{1}{4},\lambda_0,\lambda_1$
$\mathcal{N}$	3	2	3	2	4	3

[a] λ_0 is the real root of $4t^3+8t^2-8t-21=0$, while λ_1 the real root of $2t^3+2t^2+3=0$.

$$\begin{aligned}
J_m &= x_m + \sum_{l=m+1}^{2m} \alpha_{l,m} x_l + (-1)^m \theta_{m-1}(x_{2m-1} + \alpha_{2m,2m-1} x_{2m}),\\
J_{m+k} &= x_{m+k} + \sum_{l=m+k+1}^{2m} \alpha_{l,m+k} x_l + (-1)^{m+k} \theta_{m-1-k}(x_{2m-1} + \alpha_{2m,2m-1} x_{2m}) - (-1)^{m+k}\theta_{m-k} x_{2m},\\
J_{2m-2} &= x_{2m-2} + \alpha_{2m-1,2m-2} x_{2m-1} + \alpha_{2m,2m-2} x_{2m} - (m-2)\theta_2 x_{2m},\\
J_{2m-1} &= x_{2m-1} + \alpha_{2m,2m-1} x_{2m},\\
J_{2m} &= x_{2m}.
\end{aligned} \tag{19}$$

We see at once that J_{2m} is an invariant, corresponding to the generator of the center. Successive elimination of the parameters $\alpha_{j,k}$ from the equations $J_{2m-1}, J_{2m-2}, \ldots$ allows us to obtain a polynomial of degree m that depends only on the coordinates x_i.

4.1. The parameterized families

In this paragraph we determine the generalized Casimir invariants for those algebras $\mathfrak{n}_{j,\alpha}$ of proposition 1 depending on a continuous parameter. Table 1 gives the number of invariants for those having non-central Casimir operators. Clearly, for all algebras having only one invariant, the generator of the center is the Casimir operator. Since all the families have a one-dimensional center, it is only necessary to give the non-central invariants.

Lemma 2. *Let* $\mathfrak{n}_{j,\alpha}$, $j=7,\ldots,11$. *Then the non-central Casimir invariants* C_i*, where the subindex i denotes the degree, are given by*

(i) $\mathfrak{n}_{7,-1}$:

$$C_2 = x_6^2 - 2x_5x_7,$$
$$C_5 = 3x_7^3(x_2x_7 - x_3x_6 + x_4x_5) + 2x_5x_6^3x_7 - \tfrac{2}{5}x_6^5 - 3x_5^2x_6x_7^2,$$

(ii) $\mathfrak{n}_{8,\alpha}$:

$$C_4 = \alpha\left(4x_4x_8^3 - x_7^4 + 4x_6x_7^2x_8 - 4x_5x_7x_8^2 - 2x_6^2x_8^2\right) + 4x_4x_8^3 - x_7^4 - 4x_6^2x_8^2,$$

(iii) $\mathfrak{n}_{9,-2}$:

$$C_2 = 2x_7x_9 - x_8^2,$$
$$\begin{aligned}C_7 = {} & 210x_2x_9^6 - (210x_3x_8 - 70x_4x_7)x_9^5 + \left(70x_4x_8^2 + 35x_6x_7^2 - 70x_5x_7x_8\right)x_9^4\\ & - 35\left(x_7^3x_8x_9 + x_7^2x_8^3\right)x_9^2 - 14x_7x_8^5x_9 + 2x_8^7,\end{aligned}$$

(iv) $\mathfrak{n}_{9,-1}$:

$$C_2 = x_6^2 - 2x_5x_7 + 2x_4x_8 - 2x_3x_9,$$
$$C_3 = x_8^3 - 3x_7x_8x_9 + 3x_6x_9^2,$$

(v) $\mathfrak{n}_{9,\frac{1}{2}}$:

$$C_2 = 2x_7x_9 - x_8^2,$$
$$\begin{aligned}C_7 = {} & -35x_2x_9^6 - (175x_5x_6 - 35x_3x_8 - 105x_4x_7)x_9^5 - \left(70x_4x_8^2 + 210x_6x_7^2\right.\\ & \left. - 175x_6^2x_8 - 70x_5x_7x_8\right)x_9^4 + 210x_7^3x_8x_9^3 - 210x_7^2x_8^3x_9^2 + 84x_7x_8^5x_9 - 12x_8^7,\end{aligned}$$

(vi) $\mathfrak{n}_{10,\alpha}$ $(\alpha \neq -1, \frac{1}{2})$:

$$C_5(\alpha) = \tfrac{2}{5}(2+\alpha)x_9^5 - 2(2+\alpha)x_8x_9^3x_{10} + (2\alpha+5)x_8^2x_9x_{10}^2 + 2(1+\alpha)x_7x_9^2x_{10} - (2\alpha+5)x_7x_8x_{10}^3 - (2\alpha-1)x_6x_9x_{10}^3 + (2\alpha-1)x_5x_{10}^4,$$

(vii) $\mathfrak{n}_{10,-1}$:

$$C_2 = x_9^2 - 2x_8x_{10},$$
$$C_2' = x_6^2 - 2x_5x_7 + 2x_4x_8 - 2x_3x_9 + 2x_2x_{10},$$
$$C_5(1),$$

(viii) $\mathfrak{n}_{10,\frac{1}{2}}$:

$$C_2 = x_9^2 - 2x_8x_{10},$$
$$C_5(\tfrac{1}{2})$$
$$\begin{aligned} C_{10} = {} & (1120x_2x_8 - 1680x_3x_7 + 1400x_5^2)x_{10}^8 + 1680x_7^2x_9x_{10}(2x_7x_{10} - 3x_8x_9) \\ & + (1680(x_4x_7x_9 - 2x_6x_7^2) + 560(x_3x_8x_9 - x_2x_9^2) + 2800x_6^2x_8 \\ & + 1120(x_5x_7x_8 - x_4x_{10}^2))x_{10}^7 + 1260(x_7^2x_9^4 + 3x_7x_8^2x_9^2)x_{10}^4 \\ & - 168x_8x_9^4(5x_8^2 + 11x_7x_9)x_{10}^3 + 8x_9^6x_{10}^2(33x_7x_9 + 91x_8^2) \\ & - 215x_8x_9^8x_{10} + \tfrac{43}{2}x_9^{10}, \end{aligned}$$

(ix) $\mathfrak{n}_{11,-\frac{1}{4}}$:

$$\begin{aligned} C_5 = {} & 45(x_6x_{11}^4 - x_7x_{10}x_{11}^3) + 15(x_8x_{10}^2x_{11}^2 + x_8x_9x_{11}^3 - x_9^2x_{10}x_{11}^2) \\ & + 5x_9x_{10}^3x_{11} - x_{10}^5, \end{aligned}$$
$$\begin{aligned} C_6 = {} & 54(x_5x_{11}^5 + x_7x_{10}^2x_{11}^3 - x_7x_9x_{11}^4 - x_6x_{10}x_{11}^4) - 18x_8x_{10}^3x_{11}^2 + x_{10}^6 \\ & + 21x_9^2x_{10}^2x_{11}^2 - 6x_9x_{10}^4x_{11} + 27x_8^2x_{11}^4 - 14x_9^3x_{11}^3, \end{aligned}$$

(x) $\mathfrak{n}_{11,\lambda_0}$:

$$C_3 = x_{10}^3 - 3x_9x_{10}x_{11} + 3x_8x_{11}^2,$$
$$C_8 = (4\alpha+1)(2\alpha+5)(2\alpha^2+4\alpha+5)(2\alpha^3+2\alpha^2+3)^3x_3x_{11}^7 + \cdots,$$

(xi) $\mathfrak{n}_{11,\lambda_1}$:

$$C_2 = 2x_9x_{11} - x_{10}^2,$$
$$C_9 = (4\alpha+1)(2\alpha^2+4\alpha+5)(4\alpha^3+8\alpha^2-8\alpha-21)^2x_2x_{11}^8 + \cdots.$$

For C_8 and C_9 we avoid the explicit expressions of the corresponding invariant, since the first involves 340 terms and the latter 367.

5. Solvable extensions of $\mathfrak{n}_{n,i}$

By the general theory of Lie algebras, any solvable Lie algebra $\mathfrak{r}$ over the reals admits a decomposition $\mathfrak{r} = \mathfrak{t} \overrightarrow{\oplus} \mathfrak{n}$ satisfying the relations

$$[\mathfrak{t}, \mathfrak{n}] \subset \mathfrak{n}, \qquad [\mathfrak{n}, \mathfrak{n}] \subset \mathfrak{n}, \qquad [\mathfrak{t}, \mathfrak{t}] \subset \mathfrak{n}, \tag{20}$$

where $\mathfrak{n}$ is the maximal nilpotent ideal (nilradical) of $\mathfrak{r}$ and $\overrightarrow{\oplus}$ denotes the semidirect product. It was proven in [10] that the dimension of the nilradical satisfies the following inequality $\dim \mathfrak{n} \geqslant \frac{1}{2}\dim \mathfrak{r}$. It follows at once from the Jacobi identity that for any $X \in \mathfrak{t}$, the adjoint operator $\mathrm{ad}(X)$ acts as a derivation of the nilpotent algebra $\mathfrak{n}$. Given a basis $\{X_1, \ldots, X_n\}$ of $\mathfrak{t}$ and arbitrary scalars $\alpha_1, \ldots, \alpha_n \in \mathbb{R} - \{0\}$, we say that these elements are nil-independent if for $k \geqslant 1$

$$(\alpha_1 \mathrm{ad}(X_1) + \cdots + \alpha_n \mathrm{ad}(X_n))^k \neq 0, \tag{21}$$

that is, the matrix $\alpha_1 \mathrm{ad}(X_1) + \cdots + \alpha_n \mathrm{ad}(X_n)$ is not nilpotent. The nil-independence of these basis elements follows from the fact that the nilradical is a maximal nilpotent ideal of $\mathfrak{r}$ [10].

This fact imposes a first restriction on the dimension of a solvable Lie algebra having a given nilradical, namely, that $\dim \mathfrak{r}$ is bounded by the maximal number of nil-independent derivations of the nilradical. This reduces the classification of solvable Lie algebras to finding all non-equivalent extensions determined by a set of nil-independent derivations. The equivalence of extensions is considered under the transformations of the type

$$X_i \mapsto a_{ij} X_j + b_{ik} Y_k, \quad Y_k \mapsto R_{kl} Y_l, \tag{22}$$

where (a_{ij}) is an invertible $n\times n$ matrix, (b_{ik}) is an $n\times \dim\mathfrak{n}$ matrix and (R_{kl}) is an automorphism of the nilradical $\mathfrak{n}$.

Lemma 3. *Let $\mathfrak{n}$ be isomorphic to $\mathfrak{n}_{n,3}$, $\mathfrak{n}_{n,4}$, $\mathfrak{n}_{n,5}$ or $\mathfrak{n}_{n,6}$. Then an outer derivation f has the following form:*

(i) if $\mathfrak{n}\simeq\mathfrak{n}_{n,3}$, then

$$\begin{aligned}
&f(X_1) = f_1^1 X_1 + f_1^{n-3} X_{n-3} + f_1^{n-2} X_{n-2} + f_1^{n-1} X_{n-1},\\
&f(X_2) = 2 f_1^1 X_2 + \tfrac{n^2-6n+4}{(n-2)(n-3)} f_1^{n-3} X_{n-2} + \tfrac{n-4}{n-2} f_1^{n-2} X_{n-1},\\
&f(X_3) = 3\, f_1^1 X_3 + \tfrac{n-6}{n-2} f_1^{n-3} X_{n-1},\\
&f(X_k) = k\, f_1^1 X_k \ (4 \leqslant k \leqslant n);
\end{aligned}$$

(ii) if $\mathfrak{n}\simeq\mathfrak{n}_{n,4}$, then

$$\begin{aligned}
&f(X_1) = f_1^1 X_1 + f_1^2 X_2,\ \ f(X_2) = 2 f_1^1 X_2 + \textstyle\sum_{t=2}^{\frac{n-3}{2}} f_2^{2t+1} X_{2t+1} + f_2^{n-1} X_{n-1} + f_2^n X_n,\\
&f(X_k) = k\, f_1^1 X_k + \textstyle\sum_{t=2}^{\frac{n-3}{2}} f_2^{2t+1} X_{2t+1-k} + f_2^{n+2-k} X_n \ (3 \leqslant l \leqslant n-2)\\
&f(X_{n-1}) = (n-1) f_1^1 X_{n-1} + f_1^2 X_n, \quad f(X_n) = n f_1^1 X_n;
\end{aligned}$$

(iii) if $\mathfrak{n}\simeq\mathfrak{n}_{n,5}$, then

$$\begin{aligned}
&f(X_1) = f_1^1 X_1 + f_1^{n-3} X_{n-3} + f_1^{n-2} X_{n-2} + f_1^{n-1} X_{n-1},\\
&f(X_2) = 2 f_1^1 X_2 + \tfrac{n-8}{2} f_1^{n-3} X_{n-2} - \tfrac{n-4}{2} f_1^{n-2} X_{n-1},\\
&f(X_3) = 3\, f_1^1 X_3 + \tfrac{n-6}{2} f_1^{n-3} X_{n-1},\\
&f(X_k) = k\, f_1^1 X_k \ (4 \leqslant k \leqslant n);
\end{aligned}$$

(iv) if $\mathfrak{n}\simeq\mathfrak{n}_{n,6}$, then

$$\begin{aligned}
&f(X_1) = f_1^1 X_1 + f_1^{n-3} X_{n-3} + f_1^{n-2} X_{n-2},\\
&f(X_2) = 2 f_1^1 X_2 - (n-5) f_1^{n-3} X_{n-2} + f_2^n X_n,\\
&f(X_3) = 3\, f_1^1 X_3 - \tfrac{n-5}{2} f_1^{n-3} X_{n-1},\\
&f(X_k) = k\, f_1^1 X_k \ (4 \leqslant k \leqslant n).
\end{aligned}$$

Proof. We perform the explicit computations for $\mathfrak{n}_{n,4}$, the other algebras being treated similarly. For convenience we write any derivation f in the form $f(X_i) = f_i^j X_j$, where the f_i^j are scalars for any $1 \leqslant i, j \leqslant n$.

The condition $[f(X_1), X_2] + [X_1, f(X_2)] = f(X_3)$ shows that

$$f(X_3) = \left(f_1^1 + f_2^2\right) X_3 + \sum_{k=3}^{n-2} f_2^k X_{k+1} + \left(f_2^{n-1} - f_1^{n-2}\right) X_n. \tag{23}$$

Since $X_{2+t} = \mathrm{ad}(X_1)^t(X_2)$ for $t \geqslant 1$, iteration of (23) implies that

$$f(X_l) = \left((l-2) f_1^1 + f_2^2\right) X_l + \sum_{k=3}^{n+1-l} f_2^k X_{k+l-2} + \left(f_2^{n+2-l} + (-1)^{l+1} f_1^{n+1-l}\right) X_n$$

for $3 \leqslant l \leqslant n-1$ and $f(X_n) = ((n-2)f_1^1 + f_2^2)X_n$. The conditions $[f(X_2), X_{2t+1}] + [X_2, f(X_{2t+1})] = 0$ for $1 \leqslant t \leqslant \frac{n-5}{2}$ further show that $f_2^1 = 0$ and $f_2^{n-2t-1} = 0$. We now evaluate the Leibniz condition for the pair (X_2, X_{n-2}) to obtain

$$[f(X_2), X_{n-2}] + [X_2, f(X_{n-2})] = ((n-4)f_1^1 + 2f_2^2)X_n = ((n-2)f_1^1 + f_2^2)X_n,$$

from which we get that $f_2^2 = 2f_1^1$. The remaining brackets give no new conditions on the coefficients f_i^j. Thus any derivation f of $\mathfrak{n}_{n,4}$ has the form

$$\begin{aligned}
f(X_1) &= f_1^1 X_1 + \sum_{l=2}^{n} f_1^l X_l \\
f(X_2) &= 2f_1^1 X_2 + \sum_{t=2}^{\frac{n-3}{2}} f_2^{2t+1} X_{2t+1} + f_2^{n-1} X_{n-1} + f_2^n X_n \\
f(X_k) &= kf_1^1 X_k + \sum_{t=2}^{\frac{n-k}{2}} f_2^{2t+1} X_{2t+1-k} + (f_2^{n+2-k} + (-1)^k f_1^{n+1-k})X_n \ (3 \leqslant k \leqslant n-2) \\
f(X_{n-1}) &= (n-1)f_1^1 X_{n-1} + f_1^2 X_n \\
f(X_n) &= nf_1^1 X_n.
\end{aligned} \tag{24}$$

We choose the basis of the space of derivations in the following form:

$$\begin{array}{llll}
F_1^1(X_i) = \mathrm{i}X_i, & 1 \leqslant i \leqslant n & & \\
F_1^k(X_1) = X_k, & F_1^k(X_{n+1-k}) = (-1)^{n+1-k} X_n, & & 1 \leqslant k \leqslant n-1 \\
F_1^n(X_1) = X_n, & & & \\
F_2^{2t+1}(X_k) = X_{k+2t-1}, & & 2 \leqslant t \leqslant \left[\dfrac{n-3}{2}\right], & 2 \leqslant k \leqslant n+1-2t.
\end{array}$$

It is not difficult to see that the inner derivations $\mathrm{ad}(X_k)$ $(1 \leqslant k \leqslant n)$ correspond to the following operators:

$$\mathrm{ad}X_1 = F_2^3, \qquad \mathrm{ad}(X_k) = F_1^{1+k}, \qquad 2 \leqslant k \leqslant n-1.$$

Therefore there are $\frac{n+3}{2}$ outer derivations, corresponding to the operators $\{F_1^1, F_1^2, F_2^{2t+1}, F_2^{n-1}\}$ for $2 \leqslant t \leqslant \frac{n-1}{2}$. □

In particular, there is only one diagonal derivation for the Lie algebras $\mathfrak{n}_{n,i}$.

Theorem 2. *Any solvable Lie algebra with the* $\mathbb{N}$*-graded nilradical* $\mathfrak{n}_{n,3}, \mathfrak{n}_{n,4}, \mathfrak{n}_{n,5}$ *or* $\mathfrak{n}_{n,6}$ *is isomorphic to one of the following algebras:*

(i) $\mathfrak{r}_{n,3}$ $(n \geqslant 13)$:
- $[T, X_i] = i\,X_i, \quad 1 \leqslant i \leqslant n,$
- $[X_i, X_j] = (j-i)X_{i+j}, \quad 1 \leqslant i < j \leqslant n-1;$

(ii) $\mathfrak{r}_{n,4}$ $(n = 2m+2 \geqslant 8)$:
- $[T, X_i] = i\,X_i, \quad 1 \leqslant i \leqslant n,$
- $[X_1, X_j] = X_{j+1}, \quad 2 \leqslant j \leqslant n-1,$
- $[X_i, X_{n-i}] = (-1)^i X_n, \quad 2 \leqslant i \leqslant m;$

(iii) $\mathfrak{r}_{n,5}$ $(n = 2m+1 \geqslant 9)$:
- $[T, X_i] = i\,X_i, \quad 1 \leqslant i \leqslant n,$
- $[X_1, X_j] = X_{j+1}, \quad 2 \leqslant j \leqslant n-1,$
- $[X_i, X_{n-i-1}] = (-1)^i X_{n-1}, \quad 2 \leqslant i \leqslant m-1,$
- $[X_i, X_{n-i}] = (-1)^{i+1}(m-i)X_n, \quad 2 \leqslant i \leqslant m-1;$

(iv) $\mathfrak{r}_{n,6}$ $(n = 2m + 2 \geqslant 10)$:

$$[T, X_i] = i\,X_i, \quad 1 \leqslant i \leqslant n,$$
$$[X_1, X_j] = X_{j+1}, \quad 2 \leqslant j \leqslant n-1,$$
$$[X_i, X_{n-i-2}] = (-1)^i X_{n-2}, \quad 2 \leqslant i \leqslant m-1,$$
$$[X_i, X_{n-i-1}] = (-1)^i (m-i) X_{n-1}, \quad 2 \leqslant i \leqslant m-1,$$
$$[X_i, X_{n-i}] = (-1)^{i+1} \tfrac{(i-2)}{2}(2m-i-1) X_n, \quad 2 \leqslant i \leqslant m.$$

Proof. Like before, we only prove the assertion explicitly for the Lie algebra $\mathfrak{n}_{n,4}$, the remaining cases being proved in a completely analogous manner. Let $F = \alpha_1 F_1^1 + \alpha_2 F_1^2 + \sum_{k\geqslant 2} \beta_2^{2k+1} F_2^{2k+1} + \beta_2^{n-1} F_2^{n-1}$ be an outer non-nilpotent derivation of $\mathfrak{n}_{n,4}$. A scaling change allows us to suppose that $\alpha_1 = 1$. By a change of basis of the type

$$X_k' = X_k, \qquad k = 1, n-1, n,$$
$$X_k' = X_k + \sum_{t=1}^{[\frac{n-k-1}{2}]} \gamma_t X_{k+2t+1}, \qquad 2 \leqslant k \leqslant n-2$$

we can successively put to zero the coefficients $\beta_2^5, \ldots, \beta_2^7, \ldots, \beta_2^{n-1}$ and β_2^n. This reduces the derivation to $F = F_1^1 + \alpha_2 F_1^2$. Now a second change of basis of the form

$$X_1' = X_1 - \alpha_2 X_2, \qquad X_{n-1}' = X_{n-1} - \alpha_2 X_n,$$

where the remaining generators remain unchanged, allows us to further suppose that $\alpha_2 = 0$; thus F is equivalent to the diagonal derivation F_1^1. □

5.1. Parameterized families

The computation of the derivation algebras for the parameterized algebras $\mathfrak{n}_{j,\alpha}$ $(j = 7, \ldots, 11)$ follows the same general pattern. However, since the number of derivations varies for some special values of the parameter α, an explicit description would give too large a number of special cases, for which reason we omit it here. The main fact is that, like for the Lie algebras seen before, they only have one diagonal derivation f, and that any non-nilpotent derivation of $\mathfrak{n}_{j,\alpha}$ can be reduced to f by means of a change of basis.

Lemma 4. *For any $j = 7, \ldots, 11$, any outer non-nilpotent derivation of the family $\mathfrak{n}_{j,\alpha}$ is equivalent to the diagonal derivation $f(X_k) = kX_k$, $1 \leqslant k \leqslant j$.*

As a consequence of this result, it appears that imposing the condition of a $\mathbb{N}$-grading implies, with only one exception, that there is only one class of solvable extensions. The failure for this result to hold for $\mathfrak{n}_{n,1}$ is a consequence of the existence of a maximal Abelian ideal of dimension $\dim \mathfrak{n}_{n,1} - 1$ (see [25] and references therein).

6. The generalized Casimir invariants of solvable extensions

We now consider the solvable Lie algebras obtained previously and compute their generalized Casimir invariants. As done before, for the parameterized families in low dimension we will give the result in tabular form (see table 2).

Proposition 2. *Let $\mathfrak{r}$ be a solvable extension of $\mathfrak{n}_{n,3}$, $\mathfrak{n}_{n,4}$, $\mathfrak{n}_{n,5}$ or $\mathfrak{n}_{n,6}$.*

(i) For any $n + 1 \geqslant 13$ we have $\mathcal{N}(\mathfrak{r}_{n,3}) = 0$ if n is odd, and $\mathcal{N}(\mathfrak{r}_{n,3}) = 1$ if n is even. In the latter case $n = 2m$, the rational invariant is given by

$$J_1 = C_{2m,m}^{2m+2} x_{2m}^{(m-2m^2)}.$$

Table 2. Harmonics for the parameterized families.

$\mathfrak{r}$	$\mathfrak{r}_{7,2}$	$\mathfrak{r}_{8,\alpha}$	$\mathfrak{r}_{9,-2}$	$\mathfrak{r}_{9,-1}$	$\mathfrak{r}_{9,\frac{1}{2}}$	$\mathfrak{r}_{10,\alpha}$
$\mathcal{N}(\mathfrak{r})$	2	1	2	2	2	1
Inv.	$J_1 = C_2^7 x_7^{-12}$ $J_2 = C_5^7 x_7^{-30}$	$J_1 = C_4^8 x_{10}^{-28}$	$J_1 = C_3^9 x_9^{-56}$ $J_2 = C_2^9 x_9^{-16}$	$J_1 = C_3 C_2^{-2}$ $J_2 = C_2^3 x_9^{-4}$	$J_1 = C_2^3 x_9^{-4}$ $J_2 = C_7^2 C_2^{-7}$	$J_1 = C_2^2 x_{10}^{-9}$
$\mathfrak{r}$	$\mathfrak{r}_{10,-1}$	$\mathfrak{r}_{10,\frac{1}{2}}$	$\mathfrak{r}_{11,-\frac{1}{4}}$	$\mathfrak{r}_{11,\lambda_0}$	$\mathfrak{r}_{11,\lambda_1}$	
$\mathcal{N}(\mathfrak{r})$	3	3	2	2	2	
Inv.	$J_1 = C_2^2 x_{10}^{-9}$ $J_2 = C_2^5 x_{10}^{-6}$ $J_3 = (C_2')^3 C_2^{-2}$	$J_1 = C_5^5 x_{10}^{-9}$ $J_2 = C_3^2 C_2^{-3}$ $J_3 = C_{10} x_{10}^{-9}$	$J_1 = C_4^{11} x_{11}^{-50}$ $J_2 = C_6^5 C_4^{-6}$	$J_1 = C_2^{11} x_{11}^{-20}$ $J_2 = C_9^2 C_2^{-9}$	$J_1 = C_3^{11} x_{11}^{-30}$ $J_2 = C_8^3 C_3^{-8}$	

(ii) For any $n+1 = 2m+2 \geqslant 8$ the Lie algebra $\mathfrak{r}_{n,4}$ has no non-trivial invariants ($\mathcal{N}(\mathfrak{r}_{n,4}) = 0$).

(iii) For $n+1 = 2m+1 \geqslant 9$ we have $\mathcal{N}(\mathfrak{r}_{n,5}) = 1$ and the rational invariant is given by

$$J_1 = C_{2m,m}^2 x_{2m}^{1-2m}.$$

(iv) For $n+1 = 2m+4 \geqslant 10$ we have $\mathcal{N}(\mathfrak{r}_{n,6}) = 2$, the rational invariants being given by

$$J_1 = C_{n,n-2}^2 C_2^{2-n}, \qquad J_2 = C_2^n x_n^{2-2n}.$$

Proof. Using the Maurer–Cartan equations of the solvable extensions of the $\mathfrak{n}_{n,i}$, it is straightforward to verify that in all cases we have $\mathcal{N}(\mathfrak{r}_{n,i}) = \mathcal{N}(\mathfrak{n}_{n,i}) - 1$. This follows at once observing that the non-nilpotent derivation F_1^1 always acts non-trivially over the center generator X_n. Further let T be the generator of $\mathfrak{r}_{n,i}$ associated with the latter derivation. Since $[T, X_n] = nX_n$, the differential operator $\widehat{X}_n$ corresponding to the center generator is $\widehat{X}_n = -nx_n\partial_t$, from which we conclude that the invariants of the solvable algebras $\mathfrak{r}_{n,i}$ do not depend on the variable t associated with T. Thus it suffices to solve the equation $\widehat{T}F = \sum_{k=1}^n kx_k \frac{\partial F}{\partial x_k} = 0$, taking into account the Casimir operators of $\mathfrak{n}_{n,i}$ already obtained.

(i) $\mathfrak{r}_{n,3}$ ($n \geqslant 12$).
The extension has an invariant only if $n = 2m$, as follows at once from the Maurer–Cartan equations. Now, from (13) and the vanishing of $\frac{\partial F}{\partial t}$ we see that the system (3) corresponding to $\mathfrak{n}_{n,3}$ is given by

$$\widehat{X}_k F = -\sum_{l=1}^{k-1}(k-l)x_{k+l}\frac{\partial F}{\partial x_l} + \sum_{j=k+1}^{2m-j}(j-k)x_{k+j}\frac{\partial F}{\partial x_j}, \qquad k = 1,\dots,m-1$$

$$\widehat{X}_{m+k} F = -\sum_{l=1}^{m-k}(m+k-l)x_{m+k+l}\frac{\partial F}{\partial x_l}, \qquad k = 0,\dots,m-1. \tag{25}$$

It follows in particular from this system that $\frac{\partial}{\partial x_j}F = 0$ for any $j \leqslant m-1$, and the system reduces to solve the first $m-1$ equations. Starting from

$$\widehat{X}_{m-1}F = x_{2m-1}\frac{\partial F}{\partial x_m} + 2x_{2m}\frac{\partial F}{\partial x_{m+1}} = 0,$$

an iteration process always allows us to express $\frac{\partial F}{\partial x_{m+k}}$ in terms of $\frac{\partial F}{\partial x_{m+k+1}}$ for $k = 0,\dots,m-2$. This leads to (rational) functions $f_k(x_m,\dots,x_{2m})$ such that

$$\frac{\partial F}{\partial x_{m+k}} = f_k(x_m,\dots,x_{2m})\frac{\partial F}{\partial x_{2m-1}}. \tag{26}$$

By the Euler theorem for homogeneous functions, we know that for the Casimir operator $C_{2m,m}$ determined in proposition 2, the following identity holds:

$$\sum_{k=1}^{2m} x_k \frac{\partial C_{2m,m}}{\partial x_k} = \sum_{k=m}^{2m} x_k \frac{\partial C_{2m,m}}{\partial x_k} = m\, C_{2m,m}. \tag{27}$$

Taking the differential operator $\widehat{T}$ associated with the generator of the solvable extension, it is immediate that $\widehat{T}(x_{2m}) = 2mx_{2m}$, and its action on the Casimir operator $C_{2m,m}$ can be rewritten as

$$\widehat{T}(C_{2m,m}) = \sum_{k=m}^{2m} kx_k \frac{\partial C_{2m,m}}{\partial x_k} = m \sum_{k=m}^{2m} x_k \frac{\partial C_{2m,m}}{\partial x_k} + \sum_{l=m+1}^{2m} (l-m)x_l \frac{\partial C_{2m,m}}{\partial x_l}.$$

Replacing $\frac{\partial F}{\partial x_{m+k}}$ by its expression in (26), we get

$$\widehat{T}(C_{2m,m}) = m^2 C_{2m,m} + \sum_{l=m+1}^{2m-1} (l-m)x_l f_{l-m}(x_m, \ldots, x_{2m}) \frac{\partial C_{2m,m}}{\partial x_{2m-1}} + 2mx_{2m} \frac{\partial C_{2m,m}}{\partial x_{2m}}.$$

A routine but cumbersome computation using the expression of $C_{2m,m}$ further shows that

$$\sum_{l=m+1}^{2m-1} (l-m)x_l f_{l-m}(x_m, \ldots, x_{2m}) \frac{\partial C_{2m,m}}{\partial x_{2m-1}} + 2mx_{2m} \frac{\partial C_{2m,m}}{\partial x_{2m}} = m(m-1)C_{2m.m}.$$

Therefore, we obtain that $C_{2m,m}$ is a semi-invariant of the extension with

$$\widehat{T}(C_{2m,m}) = m(2m-1)C_{2m,m}.$$

Now, considering the new variables $u = x_{2m}$ and $v = C_{2m,m}$, we analyze the differential equation

$$\frac{\partial \Phi}{\partial u} + \frac{(2m-1)v}{2u} \frac{\partial \Phi}{\partial v} = 0,$$

with the general solution

$$\Phi = \eta\left(\frac{v^2}{u^{2m-1}}\right).$$

An invariant of the solvable Lie algebra $\mathfrak{r}_{n,3}$ can thus be taken as the rational function $J_1 = C_{2m,m}^2 x_{2m}^{1-2m}$.

(ii) $\mathfrak{r}_{n,5}$ $(n = 2m)$.
A similar procedure to that developed above shows that

$$\widehat{T}(x_{2m}) = 2mx_{2m}, \qquad \widehat{T}(C_{2m,m}) = m(2m-1)C_{2m,m}.$$

The rational invariant can thus be chosen as

$$J_1 = C_{2m,m}^2 x_{2m}^{1-2m}.$$

(iii) $\mathfrak{r}_{n,6}$; $(n = 2m+3)$.
For the Casimir operators x_n, C_2 and $C_{n,n-2}$ computed in proposition 2, we obtain that

$$\widehat{T}(x_n) = nx_n, \qquad \widehat{T}(C_2) = 2(n-1)C_2, \qquad \widehat{T}(C_{n,n-2}) = (n-1)(n-2)C_{n,n-2}.$$

Therefore, the invariants can be chosen as the following rational functions:

$$J_1 = C_{n,n-2}^2 C_2^{2-n}, \qquad J_2 = C_2^n x_n^{2-2n}.$$

□

7. Conclusions

We have determined all solvable Lie algebras having an $\mathbb{N}$-graded nilradical $\mathfrak{n}_{n,i}$ with maximal degree of nilpotency. This class of nilpotent algebras in particular encompasses the algebras $\mathfrak{n}_{n,1}$ and $\mathfrak{n}_{n,2}$ considered in [25] and [14]. However, the latter algebras differ structurally from the remaining $\mathbb{N}$-graded algebras, in the sense that the former have Abelian ideals of codimension 2, while the algebras $\mathfrak{n}_{n,i}$ for $i = 3, 4, 5, 6$ do not have Abelian subalgebras whose dimension exceeds $[\frac{\dim \mathfrak{n}_{n,i}}{2}]$. This explains why these algebras have such a low number of invariants. These are computed by means of the method of moving frames, following the algebraic procedure of [34]. Like in the case of $\mathfrak{n}_{n,2}$, we find that there is only one class of solvable extensions for the algebras $\mathfrak{n}_{n,i}$, determined by the derivation that is induced by the $\mathbb{N}$-grading. Since the action of this derivation is always non-zero on the center of the $\mathfrak{n}_{n,i}$, the solvable extensions only have harmonics as invariants, i.e. they do not admit classical Casimir operators.

The algebras obtained, as well as the classes of [14], have potential applications in the construction of new Hamiltonian systems by the coalgebra method [15], using an appropriate symplectic realization [16], as well as in the analysis of constant potential solutions to the Yang–Mills equations and the analytical properties of Euler equations on solvable Lie algebras [42]. An interesting question in this direction is whether from the families obtained new (gauged) WZNW models on solvable Lie algebras can be extracted [43]. Although none of the algebras $\mathfrak{n}_{n,i}$ themselves have a non-degenerate quadratic Casimir operator, they contain subalgebras having this property. For example, as follows at once from the expression of the quadratic Casimir operator of $\mathfrak{g}_{9,-1}$ and $\mathfrak{g}_{10,-1}$, the subalgebra spanned by $\{X_2, \ldots, X_9\}$ ($\{X_2, \ldots, X_{10}\}$, respectively) is self-dual, and therefore admits such models. Further, the formal similarity between $\mathfrak{n}_{n,3}$ and the family of Lie algebras considered in [43] suggests that among the lattice of subalgebras of the former further models sharing structural similarities with those of [43] can be obtained. Additional constraints like singular or null gauging may also arise from the analysis of these Lie algebras.

Finally, we should remark that in [14, 25] and in this paper, we complete the analysis of solvable Lie algebras with an $\mathbb{N}$-graded nilradical of maximal nilpotency degree. In addition, some other general classes of solvable Lie algebras and their invariants in arbitrary dimension have been classified following a similar procedure (see [14] and references therein for further details), such as

(i) solvable Lie algebras with naturally graded nilradical of nilpotency degree $n - 1$ [25, 26];
(ii) solvable Lie algebras with Abelian nilradical [22];
(iii) solvable Lie algebras with Heisenberg nilradical [36];
(iv) solvable Lie algebras with triangular nilradicals [21, 23, 28];
(v) certain classes of stable Lie algebras [24]; and
(vi) classes of solvable Lie algebras whose n-dimensional nilradicals have a nilpotency degree $n - 2$ or $n - 3$ [44, 45].

The preceding (non-exhaustive) list covers the main types of general nilradicals that are completely classified. Due to the impossibility of general classifications, in order to extend these results to other classes of solvable Lie algebras, different criteria for the choice of the nilradical should be considered. In this context, nilpotent Lie algebras that possess some grading are adequate candidates for this purpose, since they often have additional properties that are potentially useful for establishing models for different physical phenomena. One of such properties is e.g. the possibility of constructing Lie algebras adapted to a predetermined invariant metric (or the quadratic Casimir operator), which allows us to find suitable generalizations of σ-model background fields that are already known [43].

Acknowledgments

The author was partially supported by the research grants MCM2006-09152 of the Ministerio de Educación y Ciencia and GR58/4120818-920920 of the UCM-BSCH.

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